New Approach to Constrained Predictive Control with Simultaneous Model Identification

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A new framework to closed-loop process identification is proposed. It relies on simultaneous constrained model predictive control (MPC) and identification (MPCI). MPCI obtains sufficient model information on a process under constrained MPC, while minimally disturbing that process. To select a process input at each time step, MPCI solves a constrained optimization problem on-line with respect to process input values over a finite moving horizon. These process inputs should satisfy all conventional MPC constraints, as well as additional constraints that assure persistent excitation of the process by these inputs. The persistent excitation constraint can be enabled or disabled, according to process identification needs of the closed loop. An iterative scheme is proposed for the numerical solution of the on-line optimization problem. At each iteration, that iterative scheme finds a suboptimal feasible solution of the on-line optimization problem by solving a semidefinite programming problem whose global convergence is guaranteed. The effectiveness of the proposed new methodology is illustrated through simulations on a linear time-varying process.

Introduction

Model predictive control (MPC) was born from the simple idea of controlling a process by selecting a process input at each time step, after performing constrained on-line optimization. The on-line optimization problem for MPC is formulated on the basis of process measurements, a process model, and disturbance prediction over a finite moving (receding) horizon (Prett and García, 1988). For processes with constraints on process inputs and/or outputs, constrained MPC is currently the most effective methodology for treating constraints in a systematic manner.

Any model used by MPC will always be an imperfect representation of a real process. While rigorous methods exist for the design of MPC systems that guarantee robust stability and good performance in the presence of modeling inaccuracy (Rawlings and Muske, 1993; Genceli and Nikolaou, 1993, 1995; Vuthandam et al., 1995; Zheng and Morari, 1993; Michalska and Mayne, 1993), the need for the development of a model for a process under feedback control may frequently arise. This problem is known as *closed-loop identification*. Identification of a process under closed-loop MPC may be required when the performance of MPC is not acceptable and a more accurate model is needed by the MPC controller. For a linear process with a linear controller, unacceptable controller performance can be detected using statistical analysis. Examples include the use of minimum variance control (Åström, 1970) as demonstrated by Harris (1989), Stanfelj et al. (1993), and Harris et al. (1995); and hypothesis testing for internal model control (Tyler and Morari, 1995). Alternatively, closed-loop process identification may be required to prevent the closed-loop performance of MPC from being unacceptable due to poor model accuracy.

Closed-loop identification has been addressed extensively in a linear stochastic control setting (Åström and Wittenmark, 1989). Good discussions from a stochastic control viewpoint on how to circumvent the difficulty of nonidentifiability and further obtain a good model for a process under feedback control are given in Box (1976), Gustavsson et al. (1977), Caines (1984), and Melo and Friedly (1992). Van den Hof and Schrama (1994) and Gevers (1993) review recent research on new criteria for closed-loop identification of state space or input–output models for control purposes. However, in all of the literature on closed-loop identification just cited, the presence of constraints on process inputs and/or outputs has not been explicitly treated. Thus, the problem we

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address in this study is the following: *Obtain sufficient model information about a process under constrained MPC, while minimally perturbing that process.*

To address this problem we propose a new class of constrained controllers for simultaneous model predictive control and identification (MPCI). The new control strategy relies on the following idea: *To select a process input at each time step, solve on-line a constrained optimization problem with respect to process input values over a finite moving horizon.* These process inputs should satisfy all conventional MPC constraints, as well as additional constraints that assure *persistent excitation* (PE) of the process by these inputs.

In this article we apply MPCI to time-varying processes that are modeled by finite-impulse-response (FIR) models, and do not change frequently. The nontrivial extensions to deterministic autoregressive moving-average (ARMA) models, including open-loop unstable processes, are under preparation.

The rest of this article is structured as follows: we first formulate the MPCI on-line optimization problem, and then outline its solution. We illustrate MPCI with computer simulations for a linear time-varying system. Finally, we discuss further possibilities and open problems for MPCI, and propose future directions.

Formulation of Model Predictive Control and Identification

A common practice in on-line optimization-based control is to use a process model for predicting the future behavior of the process for potential manipulated input moves. If, however, the process is time-varying, then the model accuracy will get weaker, the controller performance may deteriorate, and the closed-loop may possibly become unstable. Hence, the process model may have to be updated on-line while the process is under feedback control. This on-line adaptation requires persistent excitation of the input signals (Bitmead, 1984). Our aim is to keep this persistent excitation at a sufficient level for identification while maintaining the best possible on-line objective. In this manner, the existence of persistent excitation would minimally deteriorate the closed-loop performance while allowing good parameter estimation. Next, we show how to formulate and solve such an on-line optimization problem.

Persistent excitation requirement in parameter estimation

Let the following linear time-varying model be used to describe a process at current time k

$$y(l/k) = d(l/k) + \sum_{i=1}^{n} g_i(l/k)u(l-i/k), \tag{1}$$

where

$$y(l/k) = \begin{cases} y(l) = \text{current or past measured output at time} & l \leq k \\ \quad\quad\;\; \text{predicted future output at time} & l > k \end{cases}$$

$$u(l/k) = \begin{cases} u(l) = \text{implemented past input at time} & l < k \\ \quad\quad\;\; \text{current or future potential input at time} & l \geq k \end{cases}$$

$d(l/k)$ = disturbance estimate at time l made at time k

$g_i(l/k)$ = model coefficient estimate at time l made at time k

In vector notation, Eq. 1 becomes

$$y(l/k) = \boldsymbol{\phi}(l/k)^T \boldsymbol{\theta}(l/k), \tag{2}$$

where

$$\boldsymbol{\phi}(l/k) = [u(l-1/k), u(l-2/k), \ldots, u(l-n/k), 1]^T \tag{3}$$

$$\boldsymbol{\theta}(l/k) = [g_1(l/k), g_2(l/k), \ldots, g_n(l/k), d(l/k)]^T. \tag{4}$$

Application of Eq. 2 for $l = k-m+1, \ldots, k$ yields

$$\mathbf{y}(k) = \begin{bmatrix} \boldsymbol{\phi}(k/k)^T \boldsymbol{\theta}(k/k) \\ \boldsymbol{\phi}(k-1/k)^T \boldsymbol{\theta}(k-1/k) \\ \vdots \\ \boldsymbol{\phi}(k-m+1/k)^T \boldsymbol{\theta}(k-m+1/k) \end{bmatrix}, \tag{5}$$

where

$$\mathbf{y}(k) = [y(k), y(k-1), \ldots, y(k-m+1)]^T. \tag{6}$$

Assuming that

$$\boldsymbol{\theta}(k/k) \approx \boldsymbol{\theta}(k-1/k) \approx \cdots \approx \boldsymbol{\theta}(k-m+1/k) \approx \bar{\boldsymbol{\theta}}(k), \tag{7}$$

we can estimate $\bar{\boldsymbol{\theta}}(k)$ by minimizing, with respect to $\bar{\boldsymbol{\theta}}(k)$, the weighted sum of square errors (Anderson and Johnson, 1982):

$$\begin{aligned} &\left\| \mathbf{y}(k) - \begin{bmatrix} \boldsymbol{\phi}(k/k)^T \\ \boldsymbol{\phi}(k-1/k)^T \\ \vdots \\ \boldsymbol{\phi}(k-m+1/k)^T \end{bmatrix} \bar{\boldsymbol{\theta}}(k) \right\|_\Lambda^2 \\ &= \left(\mathbf{y}(k) - \begin{bmatrix} \boldsymbol{\phi}(k/k)^T \\ \boldsymbol{\phi}(k-1/k)^T \\ \vdots \\ \boldsymbol{\phi}(k-m+1/k)^T \end{bmatrix} \bar{\boldsymbol{\theta}}(k) \right)^T \\ &\times \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda^{m-1} \end{bmatrix} \\ &\times \left(\mathbf{y}(k) - \begin{bmatrix} \boldsymbol{\phi}(k/k)^T \\ \boldsymbol{\phi}(k-1/k)^T \\ \vdots \\ \boldsymbol{\phi}(k-m+1/k)^T \end{bmatrix} \bar{\boldsymbol{\theta}}(k) \right), \end{aligned} \tag{8}$$

where $\lambda \in (0,1]$ is a forgetting factor, whose purpose is to gradually deemphasize older data. Hence, at any sampling time k, the process parameters can be approximated as

$$\theta(k/k) \approx \bar{\theta}(k) = \left[\sum_{j=0}^{m-1} \lambda^j \phi(k-j/k)\phi(k-j/k)^T\right]^{-1} \times [\phi(k/k) \quad \lambda\phi(k-1/k) \cdots \lambda^{m-1}\phi(k-m+1/k)]y(k). \tag{9}$$

Equation 9 requires that the information matrix

$$M \triangleq \sum_{j=0}^{m-1} \lambda^j \phi(k-j/k)\phi(k-j/k)^T$$

be well conditioned. The last requirement is satisfied if there exist two positive numbers ρ_0 and ρ_1 such that

$$\rho_1 I \succcurlyeq \sum_{j=0}^{m-1} \lambda^j \phi(k-j/k)\phi(k-j/k)^T \succcurlyeq \rho_0 I \tag{10}$$

where the number $\kappa \triangleq (\rho_1/\rho_0)$ provides an upper bound for the condition number of the information matrix M, and should not be excessively large. The matrix notation $A \succcurlyeq B$ means that the matrix $A - B$ is positive semidefinite. The condition, Eq. 10, is the strong persistent excitation (of order $n+1$) criterion (Goodwin and Sin, 1984).

Persistent excitation requirement in simultaneous parameter estimation and model-based control

While the preceding methodology can be directly applied for off-line parameter estimation, it may also be exploited in simultaneous on-line parameter estimation and control. The main idea we propose is the following: *At each point in time consider a finite horizon in the future and minimize an objective function over that horizon, with respect to current and future process inputs*. In addition to satisfying all standard MPC constraints, these inputs should also satisfy the *persistent excitation* criterion.

In a typical MPC setting with quadratic objective function, the preceding idea yields the following on-line optimization problem, for MPCI.

$$\min_{u(k/k),u(k+1/k),\ldots,u(k+m+n-1/k),\mu} \sum_{i=1}^{m+n} \left[w_i(y(k+i/k)-y^{sp})^2\right] + r_i \Delta u(k+i-1/k)^2 + q\mu^2 \tag{11}$$

subject to

$$u_{\max} \geq u(k+i-1/k) \geq u_{\min}, \qquad i=1,2,\ldots,m+n \tag{12}$$

$$\Delta u_{\max} \geq \Delta u(k+i-1/k) \geq -\Delta u_{\max}, \qquad i=1,2,\ldots,m+n \tag{13}$$

$$u(k+m+i/k) = u(k+i/k), \qquad i=0,1,\ldots,n-1 \tag{14}$$

$$\sum_{j=0}^{m-1} \lambda^j \phi(k-j+i/k)\phi(k-j+i/k)^T \succcurlyeq (\rho_0-\mu) I \succ 0, \qquad i=1,2,\ldots,m+n-1 \tag{15}$$

$$y(k+i/k) = \phi(k+i/k)^T \bar{\theta}(k), \qquad i=1,2,\ldots,m+n \tag{16}$$

$$\bar{\theta}(k) = \left[\sum_{j=0}^{m-1} \lambda^j \phi(k-j/k)\phi(k-j/k)^T\right]^{-1} \times [\phi(k/k) \quad \lambda\phi(k-1/k) \cdots \lambda^{m-1}\phi(k-m+1/k)]y(k), \tag{17}$$

where

$$\Delta u(k+i-1/k) \equiv u(k+i-1/k) - u(k+i-2/k).$$

Remarks

If the process inputs are bounded, that is, $u_{\min} \leq u \leq u_{\max}$, then the existence of ρ_1 in the persistent excitation inequality 10 is guaranteed. For example, it can be shown (Appendix A) that

$$\rho_1 = (n+1)\max\{u_{\min}^2, u_{\max}^2, 1\} \sum_{j=0}^{m-1} \lambda^j \tag{18}$$

satisfies the upper bound condition of inequality 10. That is the reason why the constraint 15 involves an inequality with the controller tuning parameter ρ_0 only. The value of ρ_0 is more difficult to choose. On the one hand, ρ_0 should be large (but less than ρ_1), so that the effect of noise on the quality of parameter estimation can be minimal. On the other hand, ρ_0 should be small, so that the process output does not fluctuate too much around the desired setpoint value. It is clear that if $\rho_0 = 0$, then the preceding MPCI formulation results in a standard MPC problem, because the inequalities

$$\sum_{j=0}^{m-1} \lambda^j \phi(k-j+i/k)\phi(k-j+i/k)^T \succcurlyeq 0, \qquad i=1,2,\ldots,m+n-1$$

are trivially satisfied, and need not be included in the optimization.

The selection of the value of the tuning parameter ρ_0 is a challenging problem and subject of current research.

Figure 1 shows that MPCI has an m-periodic sequence of future inputs $u(k+i/k)$, as Eq. 14 implies. This is in contrast to conventional MPC, for which future manipulated inputs are constrained to remain constant for $i \geq m$ so that the projected process output over the moving horizon can reflect the desired closed-loop behavior of the process, that is, asymptotic settling of the process output at the setpoint. MPCI, however, purports to constantly perturb the process sufficiently (but with minimum output error) for the development of a process model.

The persistent excitation condition, Eq. 15, requires that $m > n$, otherwise the information matrix M will be singular (see Appendix D).

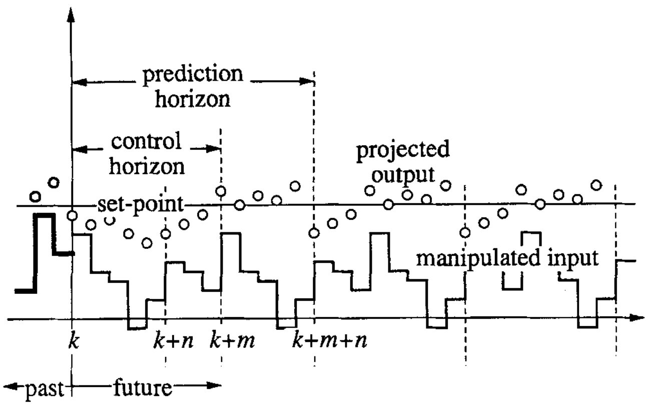

Figure 1. MPCI moving horizon.

The role of the relaxation variable μ that appears in the objective function Eq. 11 and the constraint inequality 15 is to guarantee that a feasible solution exists for the preceding optimization problem. Ideally μ should be equal to 0.

Persistent excitation is maintained in the closed loop, because of inequalities 15. Indeed, for $i = 1$, inequalities 15 imply

$$\sum_{j=0}^{m-1} \lambda^j \boldsymbol{\phi}(k-j+1/k)\boldsymbol{\phi}(k-j+1/k)^T \succcurlyeq (\rho_0 - \mu)\boldsymbol{I} \succ \mathbf{0}$$

$$\Rightarrow \sum_{j=0}^{m-1} \lambda^j [u(k-j/k),\ldots,u(k-j-n),1]^T$$

$$\times [u(k-j/k),\ldots,u(k-j-n),1] \succcurlyeq (\rho_0 - \mu)\boldsymbol{I} \succ \mathbf{0}$$

$$\Rightarrow [u(k/k),\ldots,u(k-n),1]^T [u(k/k),\ldots,u(k-n),1]$$

$$+ \sum_{j=1}^{m-1} \lambda^j [u(k-j),\ldots,u(k-j-n),1]^T$$

$$\times [u(k-j),\ldots,u(k-j-n),1] \succcurlyeq (\rho_0 - \mu)\boldsymbol{I} \succ \mathbf{0}.$$

This inequality implies that the current input $u(k/k)$ (to be selected at current time k) will have to satisfy the PE condition.

Output constraints can be added to the preceding optimization problem. To simplify the ensuing solution methodology, we presently consider only input constraints.

Equation 17 provides the weighted least-squares (Eq. 8) estimate of the model parameters and disturbance vector $\bar{\boldsymbol{\theta}}(k)$. $\bar{\boldsymbol{\theta}}(k)$ is assumed to remain constant throughout the optimization horizon.

To prevent sharp changes in $\bar{\boldsymbol{\theta}}(k)$ one may consider alternatives to weighted least-squares minimization (Eq. 8), such as constrained weighted least-squares minimization, or weighted least-squares minimization with an additional penalty on $\|\bar{\boldsymbol{\theta}}(k) - \bar{\boldsymbol{\theta}}(k-1)\|_2^2$.

The selection of the excitation window and control horizon length m is not trivial. For good control and large projected signal-to-noise ratio at each time k, we would like large m. However, for a large m the assumption that the process is quasi time invariant over the horizon might pose problems, when the process is significantly time varying. The selection of the m value is still an open issue. In the example presented in this article we used the rules of thumb $p = 3n$ and $m = p - n = 2n$.

Solution of the MPCI On-Line Optimization Problem

Linear matrix inequalities

A linear matrix inequality (LMI) has the form

$$\boldsymbol{W}_0 + \sum_{i=1}^{K} z_i \boldsymbol{W}_i \triangleq \boldsymbol{W}(z) \succcurlyeq \mathbf{0} \tag{19}$$

where $z \in \Re^K$ is a variable, and the symmetric matrices $\boldsymbol{W}_i = \boldsymbol{W}_i^T \in \Re^{N\times N}$, $i = 0, \ldots, K$, are given. The inequality symbol in Eq. 19 means that $\boldsymbol{W}(z)$ is a positive semidefinite matrix. In the sequel, the inequality symbol will denote matrix definiteness whenever the inequality involves matrices on both sides.

Semidefinite programming

Semidefinite programming (SP) is a generalization of linear programming (LP) that includes an LMI constraint, as follows:

$$\min_z \boldsymbol{c}^T z, \tag{20}$$

subject to

$$\boldsymbol{W}_0 + \sum_{i=1}^{K} z_i \boldsymbol{W}_i \triangleq \boldsymbol{W}(z) \succcurlyeq \mathbf{0}. \tag{21}$$

If $\boldsymbol{W}(z)$ is a diagonal matrix, the preceding SP problem becomes an LP problem. Powerful convex programming techniques are available for efficient numerical solution of SP problems. The ellipsoid algorithm is one such algorithm that is guaranteed to solve standard SP problems in polynomial time. In practice, interior-point algorithms are known to be much more efficient (Boyd et al., 1994). Details on SP can be found in Vandenberghe and Boyd (1994).

SP formulation of the least-squares problem with LMI constraint

In this section we show how to transform a least-squares problem with linear and LMI constraints into an SP problem. Consider the minimization

$$\min_x (\boldsymbol{Ex} + \boldsymbol{f})^T (\boldsymbol{Ex} + \boldsymbol{f}), \tag{22}$$

subject to

$$\boldsymbol{Ax} \geq \boldsymbol{b} \tag{23}$$

$$\boldsymbol{F}_0 + \sum_{i=1}^{L} x_i \boldsymbol{F}_i \succcurlyeq \mathbf{0}, \tag{24}$$

where $\boldsymbol{x} = [x_1, x_2, \ldots, x_L]^T$, $\boldsymbol{f}$, $\boldsymbol{b}$ are column vectors; $\boldsymbol{F}_i$, $i = 0, 1, \ldots, L$ are symmetric matrices; $\boldsymbol{E}$, $\boldsymbol{A}$ are matrices of appropriate dimensions. The previous optimization problem contains standard least-squares objective and constraints (Eqs. 22, 23, respectively) and an additional LMI constraint (Eq. 24). This problem cannot be solved by quadratic programming (QP) algorithms. However, this is a convex optimization problem that can be reformulated as an SP problem as follows.

First, one can use the equivalence

$$(Ex+f)^T(Ex+f) \le \sigma \Leftrightarrow \begin{bmatrix} I & (Ex+f) \\ (Ex+f)^T & \sigma \end{bmatrix} \succcurlyeq 0 \tag{25}$$

(see proof in Appendix B) to convert the preceding optimization problem (Eqs. 22 to 24) to the following problem:

$$\min_{x,\sigma} \sigma, \tag{26}$$

subject to

$$Ax \ge b \tag{27}$$

$$\begin{bmatrix} I & (Ex+f) \\ (Ex+f)^T & \sigma \end{bmatrix} \succcurlyeq 0 \tag{28}$$

$$F_0 + \sum_{i=1}^{L} x_i F_i \succcurlyeq 0. \tag{29}$$

Next, the constraint Eq. 27 can be put in LMI form as follows. Let the matrix A be written as

$$A = [a_1 \quad a_2 \cdots a_L], \tag{30}$$

where a_i, $i=1, \dots, L$ are column vectors, and let the operator D be defined as

$$D\left(\begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \end{bmatrix}\right) \triangleq \begin{bmatrix} \nu_1 & 0 & \cdots \\ 0 & \nu_2 & \\ \vdots & & \ddots \end{bmatrix}.$$

Then the constraint Eq. 27 can be written as

$$-D(b) + \sum_{i=1}^{L} x_i D(a_i) \succcurlyeq 0 \tag{31}$$

For the final step, let the matrix E be written as

$$E = [e_1 \quad e_2 \quad \cdots \quad e_L], \tag{32}$$

where e_i, $i=1, \dots, L$ are column vectors. Then, the least-squares problem Eqs. 26–29 becomes the following SP problem:

$$\min_{x,\sigma} \sigma, \tag{33}$$

subject to the LMI constraint

$$\underbrace{\begin{bmatrix} \begin{bmatrix} I & f \\ f^T & 0 \end{bmatrix} & 0 & 0 \\ 0 & -D(b) & 0 \\ 0 & 0 & F_0 \end{bmatrix}}_{W_0} + \sigma \underbrace{\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{W_{L+1}} + \sum_{i=1}^{L} x_i \underbrace{\begin{bmatrix} \begin{bmatrix} 0 & e_i \\ e_i^T & 0 \end{bmatrix} & 0 & 0 \\ 0 & D(a_i) & 0 \\ 0 & 0 & F_i \end{bmatrix}}_{W_i,\, i=1,\dots,L} \succcurlyeq 0. \tag{34}$$

Equation 34 is an instance of Eq. 19 with $K = L+1$, $z = [x_1, x_2, \dots, x_L, \sigma]^T$ and corresponding matrices W_i, as shown in Eq. 34.

Sequential semidefinite programming formulation of the MPCI on-line optimization problem

Unlike standard MPC, which requires the on-line solution of a QP or LP problem, the preceding formulation of MPCI (Eqs. 11 to 17) is not a QP or LP problem. It is rather a constrained optimization problem involving the nonconvex matrix inequalities 15. We show below how one can find a suboptimal solution of that problem by solving an SP problem. In the next section we show how to use SP in an iterative methodology that searches for a local optimum.

We start from the following relation, which is always true:

$$\sum_{j=0}^{m-1} \lambda^j [\phi(k-j+i/k) - \phi^*(k-j+i/k)] \times [\phi(k-j+i/k) - \phi^*(k-j+i/k)]^T \succcurlyeq 0, \tag{35}$$

where ϕ^* is an arbitrary column vector in $\Re^{n+1}$. Rearranging terms yields

$$\begin{aligned} \sum_{j=0}^{m-1} \lambda^j \phi(k-j+i/k)\phi(k-j+i/k)^T & \\ \succcurlyeq \sum_{j=0}^{m-1} \lambda^j \phi^*(k-j+i/k)\phi(k-j+i/k)^T & \\ + \sum_{j=0}^{m-1} \lambda^j \phi(k-j+i/k)\phi^*(k-j+i/k)^T & \\ - \sum_{j=0}^{m-1} \lambda^j \phi^*(k-j+i/k)\phi^*(k-j+i/k)^T. & \end{aligned} \tag{36}$$

Hence, satisfaction of the inequality

$$\sum_{j=0}^{m-1} \lambda^j \phi^*(k-j+i/k)\phi(k-j+i/k)^T + \sum_{j=0}^{m-1} \lambda^j \phi(k-j+i/k)\phi^*(k-j+i/k)^T$$

$$-\sum_{j=0}^{m-1} \lambda^j \boldsymbol{\phi}^*(k-j+i/k)\boldsymbol{\phi}^*(k-j+i/k)^T \succcurlyeq (\rho_0-\mu)\boldsymbol{I},$$

$$i=1,2,\ldots,m+n-1 \quad (37)$$

will always guarantee satisfaction of the constraint Eq. 15. But the preceding inequality, Eq. 37, is an LMI (see Appendix C). Thus, the proposed MPCI on-line optimization, Eqs. 11 to 17 with the constraint Eq. 15 replaced by the constraint Eq. 37, is in the general form of the optimization problem Eqs. 22–24, where the optimization variable x is

$$\boldsymbol{x}=[u(k/k) \quad u(k+1/k) \quad \cdots \quad u(k+m-1/k), \mu].$$

This problem, in turn, can be easily transformed into an SP problem in the form of Eqs. 33–34, as shown earlier.

Remark. The vector $\boldsymbol{\phi}^*$ can be thought of as a point of approximate linearization of the quantity

$$\sum_{j=0}^{m-1} \lambda^j \boldsymbol{\phi}(k-j+i/k)\boldsymbol{\phi}(k-j+i/k)^T,$$

because for $\boldsymbol{\phi} \approx \boldsymbol{\phi}^*$ we have

$$\sum_{j=0}^{m-1} \lambda^j \boldsymbol{\phi}(k-j+i/k)\boldsymbol{\phi}(k-j+i/k)^T$$
$$\approx \sum_{j=0}^{m-1} \lambda^j \boldsymbol{\phi}^*(k-j+i/k)\boldsymbol{\phi}(k-j+i/k)^T$$
$$+\sum_{j=0}^{m-1} \lambda^j \boldsymbol{\phi}(k-j+i/k)\boldsymbol{\phi}^*(k-j+i/k)^T$$
$$-\sum_{j=0}^{m-1} \lambda^j \boldsymbol{\phi}^*(k-j+i/k)\boldsymbol{\phi}^*(k-j+i/k)^T, \quad (38)$$

while the lefthand side of the previous equality is greater than or equal to the righthand side because of Eq. 36.

Iterative SP solution of the MPCI on-line optimization problem

If $\boldsymbol{\phi}^*$ is poorly chosen, then the solution of the on-line SP problem, Eqs. 11–14, 37, 16, 17, may be far from the solution of the original MPCI on-line nonconvex optimization problem, Eqs. 11 to 17. To improve the selection of $\boldsymbol{\phi}^*$, we can apply an iterative procedure at time k, as follows:

Step 1. Select an initial set $\{\boldsymbol{\phi}^*(k-j+i/k),\ j=0,\ \ldots,\ m-1,\ i=1,\ \ldots,\ m+n-1\}$. A good choice is

$$\boldsymbol{\phi}^*(k-j+i/k)=[\alpha_1 \ \cdots \ \alpha_\nu \ \ 1]^T,$$

where

$$\alpha_l=\begin{cases} u(k-j+i-l) & \\ \quad \text{if } j-i+l\geq 1 & \text{(past implemented } u) \\ u_{\text{opt}}(k-j+i-l/k-1) & \\ \quad \text{if } j-i+l<1 & \begin{pmatrix}\text{optimal } u \text{ computed at} \\ k-1, \text{ but not implemented}\end{pmatrix}. \end{cases}$$

Step 2. Solve the SP problem Eqs. 11–14, 16, 17 and 37. Let the optimal solution be $\{u_{\text{opt}}(k/k),\ \ldots,\ u_{\text{opt}}(k+m-1/k), \mu_{\text{opt}}\}$.

Step 3. Update $\boldsymbol{\phi}^*$ by setting

$$\boldsymbol{\phi}^*_{\text{new}}(k-j+i/k)=[\beta_1 \ \cdots \ \beta_\nu \ \ 1]^T,$$
$$j=0,\ldots,m-1, \qquad i=1,\ldots,m+n-1,$$

where

$$\beta_l=\begin{cases} u(k-j+i-l) & \text{if } j-i+l\geq 1 \\ u_{\text{opt}}(k-j+i-l/k) & \text{if } j-i+l<1. \end{cases}$$

Step 4. If a termination criterion such as $\phi^*_{\text{new}}=\phi^*_{\text{old}}$ is satisfied, then stop. Else, go to step 2.

Remark. Even if a local minimum is reached by the iterative scheme proposed here, MPCI will not fail, since all constraints (Eqs. 12–17) will be satisfied. Of course, failure to reach the global minimum will affect the performance of the closed loop. The convergence properties of the preceding algorithm are the subject of current investigation.

MPCI startup procedure

Assume that at time $k=1$ it is desired to turn on MPCI for a certain process. As in any adaptive control scheme, a certain initial time has to be allowed before starting the on-line parameter estimation. During this transition period, a necessary amount of data is accumulated for starting on-line identification at the end of this period. Identification can then be performed at each subsequent time step continuously. We will call this first transition period the MPCI startup procedure. During the startup procedure, MPCI is implemented at each time $k=1,\ \ldots,\ m+n-1$ as an on-line optimization problem comprising Eqs. 11 to 14 along with the following modifications of Eqs. 15 and 16:

$$\sum_{j=0}^{m-1} \lambda^j \boldsymbol{\phi}(k-j+i/k)\boldsymbol{\phi}(k-j+i/k)^T \succcurlyeq (\rho_0-\mu)\boldsymbol{I},$$

$$i=m+n-k,\ldots,m+n-1 \quad (39)$$

$$y(k+i/k)=\boldsymbol{\phi}(k+i/k)^T\begin{bmatrix} g_1(k+i/1) \\ \vdots \\ g_n(k+i/1) \\ d(k+i/k)\end{bmatrix}, \quad (40)$$

where the output additive disturbance $d(k+i/k)$ is predicted according to conventional MPC as

$$d(k+i/k) = y(k/k) - \sum_{j=1}^{n} g_j(k+i/1)u(k-j/k). \quad (41)$$

Remarks

The reason for requiring the startup procedure to last from $k=1$ to $k=m+n-1$ is that the information matrix

$$\sum_{j=0}^{m-1} \lambda^j \phi(k-j/k)\phi(k-j/k)^T$$

in Eq. 17 must be invertible. That information matrix is a function of $m+n$ past inputs, from $u(k-m-n+1)$ to $u(k-1)$ for any k, as seen from Eq. 3. Therefore, at least $m+n$ past inputs must have been produced by the MPCI algorithm before the invertibility of the information matrix can be guaranteed. This implies $k \geq m+n$.

The reason for restricting the range of the index i from $m+n-k$ to $m+n-1$ in Eq. 39 instead of 1 to $m+n-1$ as in Eq. 15 is to prevent Eq. 15 from demanding excessively large moves from the input u. If the index i were left to vary from 1 to $m+n-1$, then inputs implemented at times $k \leq 0$ could adversely affect the information matrix

$$\sum_{j=0}^{m-1} \lambda^j \phi(k-j/k)\phi(k-j/k)^T$$

that appears in Eq. 15, since these inputs were not chosen by MPCI, hence were not required to satisfy a persistent excitation constraint.

The meaning of Eq. 40 is that the model

$$y(k+i/k) = d(k+i/k) + \sum_{j=1}^{n} g_j(k+i/1)u(k+i-j/k) \quad (42)$$

with fixed coefficients $g_j(k+i/1)$ is used in the on-line optimization, and no adaptation is performed, since Eq. 17 is not included in the constraints. At $k=m+n$, simultaneous identification and control, that is, Eqs. 11 to 17, resumes, and is continued beyond this point.

Illustrative Example

Let us assume that the real behavior of a linear process is described by the equation

$$y(k) = u(k-1) + 0.5u(k-2) + 0.2u(k-3) + 0.1u(k-4) + d(k) + c(k), \quad (43)$$

where d is a deterministic disturbance and c is white noise with zero mean and standard deviation equal to 0.01. We will compare two alternative controllers, MPC and MPCI, with parameters shown in Table 1. The values of the tuning parameters λ, q, and ρ_0 were determined by simulation trial and error.

The process input u must satisfy the constraints

$$-0.4 \leq u(k) \leq 0.5$$

at all times k.

Assume that the linear model

$$y(k+i/k) = 1.1u(k+i-1/k) + 0.55u(k+i-2/k) + 0.22u(k+i-3/k) + 0.11u(k+i-4/k) + d(k+i/k) \quad (44)$$

is available for the preceding process from previous data.

For MPC the disturbance is predicted as

$$d(k+i/k) = d(k/k) = y(k) - 1.1u(k-1) - 0.55u(k-2) - 0.22u(k-3) - 0.11u(k-4). \quad (45)$$

Remark. Conventional MPC will use Eq. 45 even if the process varies unless a new process model is externally supplied. MPCI, on the other hand, will use Eq. 45 only during the startup procedure, $k < m+n$, and will rely on Eq. 17 afterwards.

The process is upset by a step setpoint change

$$y^{sp} = -0.5$$

at time $k=0$. We then distinguish two time periods:

Period 1. ($0 \leq k \leq 15$). The system is upset by the step disturbance

$$d(k) = 0.1$$

implemented at time $k=0$.

Period 2. ($15 \leq k$). At $k=15$ the real process has changed as follows.

$$y(k) = -0.3u(k-1) - 0.2u(k-2) - 0.1u(k-3) + 0.05u(k-4) + d(k). \quad (46)$$

Table 1

Parameter	Symbol	Value for MPC	Value for MPCI
Control horizon length	m	8	8
Optimization horizon length	$p=m+n$	12	12
Output weighting factors	w_i	1, $i=1,\ldots,m+n$	1, $i=1,\ldots,m+n$
Move suppression term coefficients	r_i	0.1, $i=0,\ldots,m-1$ 0, $i=m,\ldots,m+n-1$	0.1, $i=0,\ldots,m+n-1$
Forgetting factor	λ	Not applicable	1
LMI softening weight	q	Not applicable	5
Persistent excitation lower bound	ρ_0	Not applicable	0.01

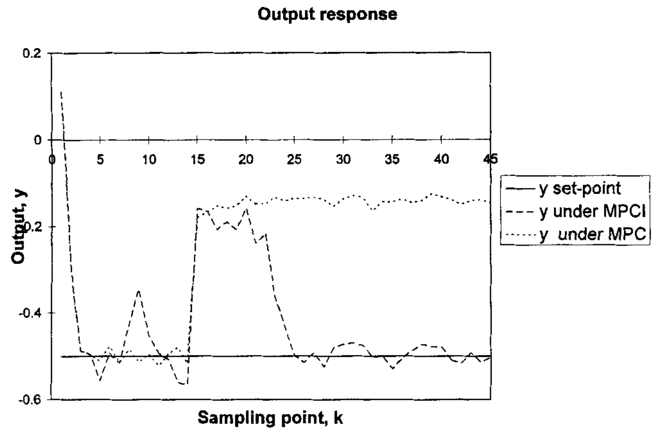

Figure 2. Comparison of process output responses under MPCI and MPC.

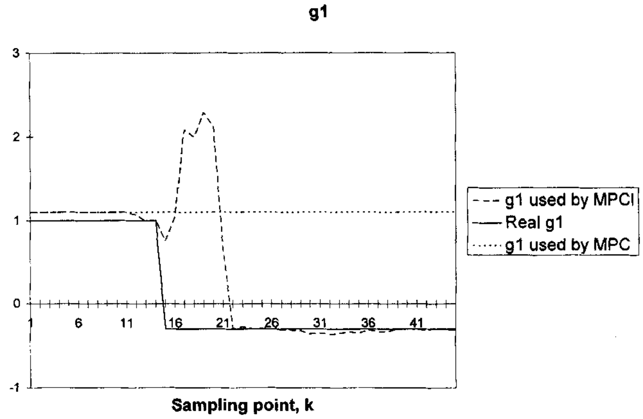

Figure 4. Comparison of process model coefficients g_1 used by MPCI and MPC.

The system is upset by the step disturbance

$$d(k) = -0.4.$$

Remark. Notice that the sign of the steady-state gain of the process has changed at $k = 15$. This is a rather extreme case of a time-varying process, that may appear in practice in systems such as autocatalytic reactors. This situation clearly illustrates the effectiveness of MPCI.

Our simulations show that conventional MPC resulted in saturation of the process input and never recovered, as illustrated in Figures 2 and 3. However, MPCI correctly identified the new plant parameters and disturbance after sufficient data were collected, that is, when $k > 21$ (Figures 4 to 8) and closely tracked the setpoint.

Conclusions and Future Research Directions

A new approach to simultaneous constrained MPC and identification (MPCI) was proposed. The new approach relies on on-line optimization of a conventional quadratic objective function over a moving horizon, with respect to process inputs that satisfy a persistent excitation constraint in addition to all conventional MPC constraints. This approach attempts to keep control performance deterioration at a minimum while performing closed-loop identification. An iterative scheme was proposed for the numerical solution of the on-line optimization problem. At each iteration, that iterative scheme finds a suboptimal feasible solution of the on-line op-

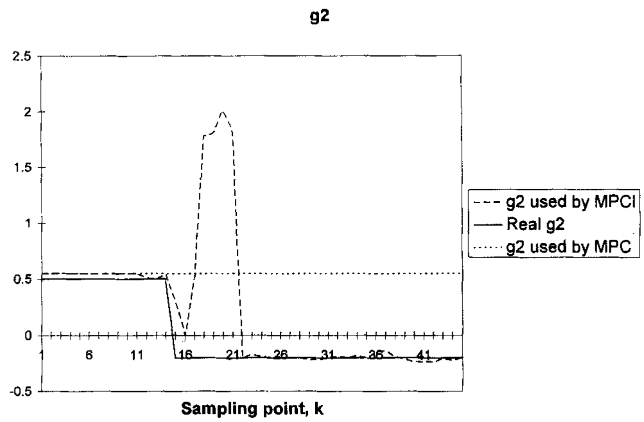

Figure 5. Comparison of process model coefficients g_2 used by MPCI and MPC.

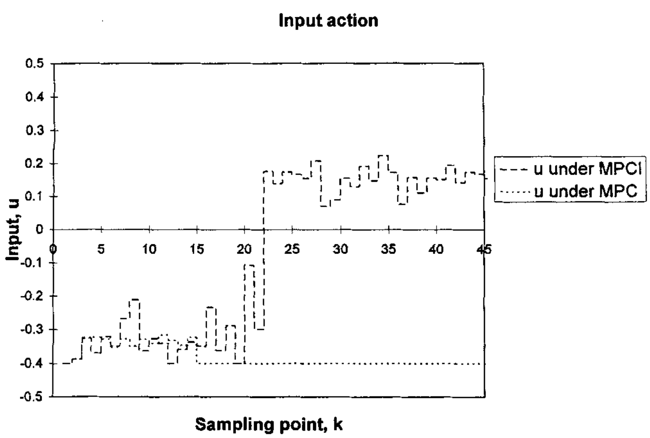

Figure 3. Comparison of process input actions under MPCI and MPC.

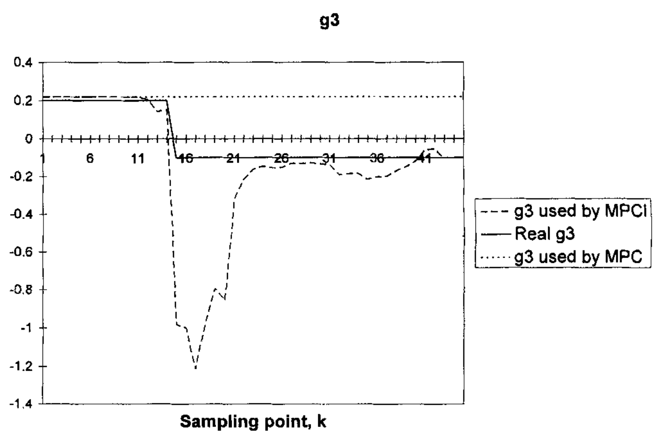

Figure 6. Comparison of process model coefficients g_3 used by MPCI and MPC.

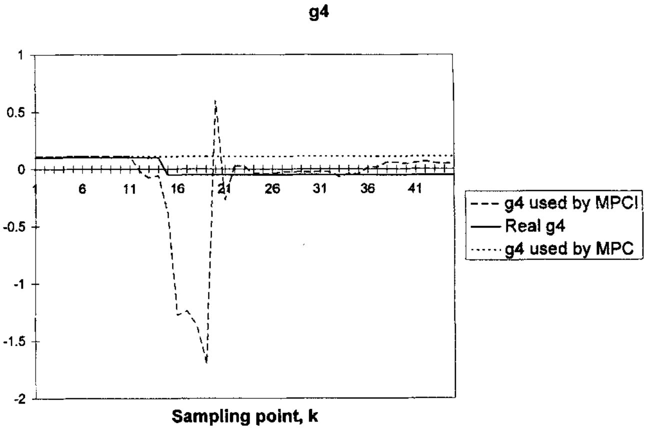

Figure 7. Comparison of process model coefficients g_4 used by MPCI and MPC.

timization problem by solving a semidefinite programming problem whose global convergence is guaranteed. An illustrative example demonstrated the applicability of the method.

A number of theoretical questions as well as variations on the proposed approach may be considered. A representative list follows:

- Closed-loop stability, performance, and robustness. These closed-loop properties depend on the design of the MPCI controller. This involves: selection of values for MPCI parameters (m, ρ_0, r_i, w_i, q, λ); the way model adaptation is performed.
- Convergence of the on-line optimization algorithm. Global convergence is guaranteed for each step of the proposed iterative scheme, because of convexity of semidefinite programming. However, convergence of the entire iterative process to a global optimum over the optimization domain needs to be examined.
- Efficiency of the on-line optimization algorithm. For example, interior-point methods are known to be almost as efficient as the simplex method for linear-programming problems of comparable size.
- Inclusion of output constraints in the on-line optimization.
- Satisfaction by process inputs of additional constraints related to good process identification, such as frequency characteristics, etc.
- Extension to multivariable systems.
- Extension to deterministic ARMA models.
- Extension to nonlinear systems.

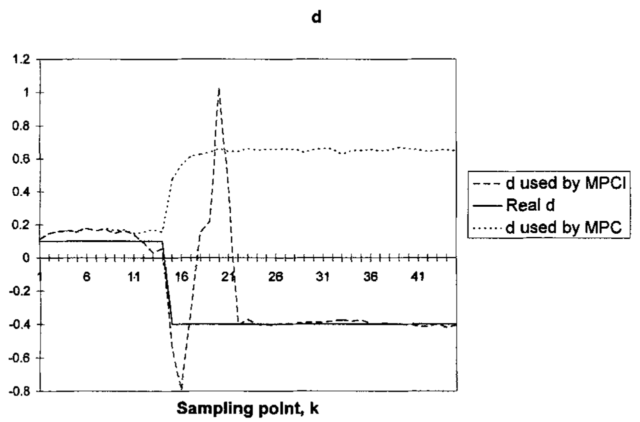

Figure 8. Comparison of unmeasured process disturbance estimated by MPCI and MPC.

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Notation

d = output additive disturbance
g = time-varying unit-pulse-response-model coefficients
n = number of pulse-response coefficients in process model
u = process-manipulated input
y = process output
$\boldsymbol{\phi}$ = regression vector whose entries are process inputs
$\boldsymbol{\theta}$ = parameter estimate vector for model parameters and disturbance
ρ_1 = upper bound of persistent excitation

Operators

$A \succcurlyeq B$ = the square matrix A-B is positive semidefinite
$A \succ B$ = the square matrix A-B is positive definite
$a \geq b$ = set of inequalities $a_i \geq b_i$ for the components of the a and b

Subscripts and superscript

max = maximum value
min = minimum value
T = transpose

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Appendix A: Proof of Existence of ρ_1 in Eq. 10 for $u_{\min} \le u \le u_{\max}$

We have that $\forall x \in \Re^{n+1}$

$$\frac{x^T \sum_{j=0}^{m-1} \lambda^j \phi(k-j/k)\phi(k-j/k)^T x}{x^T x} = \frac{\sum_{j=0}^{m-1} \lambda^j \left(\sum_{i=1}^{n} u(k-j-i/k) x_i + x_{n+1} \right)^2}{\sum_{i=1}^{n+1} x_i^2}$$

$$\le \frac{\sum_{j=0}^{m-1} \lambda^j (n+1) \left(\sum_{i=1}^{n} u(k-j-i/k)^2 x_i^2 + x_{n+1}^2 \right)}{\sum_{i=1}^{n+1} x_i^2}$$

$$\le \frac{\left((n+1) \sum_{j=0}^{m-1} \lambda^j \right) \left(\sum_{i=1}^{n} U^2 x_i^2 + x_{n+1}^2 \right)}{\sum_{i=1}^{n+1} x_i^2}$$

$$\le \left((n+1) \sum_{j=0}^{m-1} \lambda^j \right) \max_{x_1, \ldots, x_{n+1}} \frac{U^2 \sum_{i=1}^{n} x_i^2 + x_{n+1}^2}{\sum_{i=1}^{n+1} x_i^2}$$

$$= \left((n+1) \sum_{j=0}^{m-1} \lambda^j \right) \max_{x_1, \ldots, x_{n+1}} \left\{ U^2 + \frac{(1-U^2) x_{n+1}^2}{\sum_{i=1}^{n+1} x_i^2} \right\}$$

$$= \left\{ \begin{array}{ll} (n+1) U^2 \sum_{j=0}^{m-1} \lambda^j & \text{if } U \ge 1 \\ (n+1) \sum_{j=0}^{m-1} \lambda^j & \text{if } U < 1 \end{array} \right\} \triangleq \rho_1, \quad \text{(A1)}$$

where $U \triangleq \max(|u_{\max}|, |u_{\min}|)$. Equation A1 implies that

$$\rho_1 x^T x \ge \sum_{j=0}^{m-1} \lambda^j x^T \phi(k-j/k)\phi(k-j/k)^T x, \quad \forall x \in \Re^{n+1}$$

$$\Leftrightarrow \rho_1 I \succeq \sum_{j=0}^{m-1} \lambda^j \phi(k-j/k)\phi(k-j/k)^T.$$

O.E.Δ.

Appendix B: Proof of Eq. 25

The proof of the relation

$$(Ex+f)^T(Ex+f) \le \sigma \Leftrightarrow \begin{bmatrix} I & (Ex+f) \\ (Ex+f)^T & \sigma \end{bmatrix} \succeq 0$$

is given in two parts.

Forward Direction

$$(Ex+f)^T(Ex+f) \le \sigma$$
$$\Rightarrow \nu_2^T (Ex+f)^T (Ex+f) \nu_2 \le \nu_2^T \sigma \nu_2, \quad \forall \nu_2 \in \Re^1. \quad \text{(B1)}$$

The following relation always holds.

$$0 \le (\nu_1 + (Ex+f)\nu_2)^T (\nu_1 + (Ex+f)\nu_2), \quad \forall \nu_2 \in \Re^1, \quad \forall \nu_1 \in \Re^k$$

$$\Updownarrow$$

$$0 \le \nu_1^T \nu_1 + \nu_1^T (Ex+f)\nu_2 + \nu_2^T (Ex+f)^T \nu_1 + \nu_2^T (Ex+f)^T (Ex+f)^T (Ex+f) \nu_2, \quad \forall \nu_2 \in \Re^1, \quad \forall \nu_1 \in \Re^k. \quad \text{(B2)}$$

Inequalities (Eqs. B1 and B2) imply

$$0 \le \nu_1^T \nu_1 + \nu_1^T (Ex+f)\nu_2 + \nu_2^T (Ex+f)^T \nu_1 + \nu_2^T \sigma \nu_2, \quad \forall \nu_2 \in \Re^1, \quad \forall \nu_1 \in \Re^k$$

$$\Rightarrow \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix}^T \begin{bmatrix} I & (Ex+f) \\ (Ex+f)^T & \sigma \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \ge 0,$$

$$\forall \nu_2 \in \Re^1, \quad \forall \nu_1 \in \Re^k \Rightarrow \begin{bmatrix} I & (Ex+f) \\ (Ex+f)^T & \sigma \end{bmatrix} \succeq 0$$

Backward Direction

$$\begin{bmatrix} I & (Ex+f) \\ (Ex+f)^T & \sigma \end{bmatrix} \succcurlyeq 0$$

$$\Rightarrow \begin{bmatrix} (-Ex-f) \\ 1 \end{bmatrix}^T \begin{bmatrix} I & (Ex+f) \\ (Ex+f)^T & \sigma \end{bmatrix} \begin{bmatrix} (-Ex-f) \\ 1 \end{bmatrix} \geq 0$$

$$\Rightarrow (Ex+f)^T(Ex+f) \leq \sigma .$$

O.E.Δ.

Appendix C: Proof that Eq. 37 Can Be Brought into the Standard LMI Form Eq. 19

The vector $\phi(k-j+i/k)$ in the lefthand side of Eq. 37 can be written as a linear combination of the basis vectors

$$\epsilon_l \equiv \underbrace{[0 \ \cdots \ 010 \ \cdots \ 0]}_{1 \text{ in } l\text{th location}}{}^T, \qquad l = 1, \ldots, n+1 \tag{C1}$$

as

$$\phi(k-j+i/k) = \sum_{l=1}^{n} u(k-j+i-l/k)\epsilon_l + \epsilon_{n+1}. \tag{C2}$$

Therefore, for a fixed $i = 1, \ldots, m+n-1$, Eq. 37 is equivalent to

$$\begin{aligned} &\sum_{j=0}^{m-1} \lambda^j \phi^*(k-j+i/k)\phi(k-j+i/k)^T \\ &\quad + \sum_{j=0}^{m-1} \lambda^j \phi(k-j+i/k)\phi^*(k-j+i/k)^T \\ &- \sum_{j=0}^{m-1} \lambda^j \phi^*(k-j+i/k)\phi^*(k-j+i/k)^T - (\rho_0 - \mu)I \succcurlyeq 0 \\ &\Leftrightarrow \sum_{j=0}^{m-1} \lambda^j \phi^*(k-j+i/k)\left[\sum_{l=1}^{n} u(k-j+i-l/k)\epsilon_l^T + \epsilon_{n+1}^T\right] \\ &+ \sum_{j=0}^{m-1} \lambda^j \left[\sum_{l=1}^{n} u(k-j+i-l/k)\epsilon_l + \epsilon_{n+1}\right]\phi^*(k-j+i/k)^T \\ &\qquad - Q_i - (\rho_0 - \mu)I \succcurlyeq 0, \end{aligned} \tag{C3}$$

where

$$Q_i \triangleq \sum_{j=0}^{m-1} \lambda^j \phi^*(k-j+i/k)\phi^*(k-j+i/k)^T. \tag{C4}$$

Equation C3 is equivalent to

$$\begin{aligned} &\sum_{j=0}^{m-1}\sum_{l=1}^{n} u(k-j+i-l/k)M_{i,j,l} + N_i \\ &+ \sum_{j=0}^{m-1}\sum_{l=1}^{n} u(k-j+i-l/k)M_{i,j,l}^T + N_i^T - Q_i - \rho_0 I + \mu I \succcurlyeq 0, \end{aligned} \tag{C5}$$

where

$$M_{i,j,l} \triangleq \lambda^j \phi^*(k-j+i/k)\epsilon_l^T \tag{C6}$$

$$N_i \triangleq \sum_{j=0}^{m-1} \lambda^j \phi^*(k-j+i/k)\epsilon_{n+1}^T. \tag{C7}$$

Using the index transformation

$$s = -j + i - l \tag{C8}$$

in Eq. C4, we get

$$\begin{aligned} &\sum_{s=i+1-m-n}^{i-1} u(k+s/k) \sum_{j=\max\{0,i-s-n\}}^{\min\{m-1,i-s-1\}} \\ &\quad \times \left[M_{i,j,i-j-s} + M_{i,j,i-j-s}^T\right] + \mu I + N_i + N_i^T - Q_i - \rho_0 I \\ &\equiv \sum_{s=i+1-m-n}^{i-1} u(k+s/k)R_{i,s} + \mu I + N_i + N_i^T - Q_i - \rho_0 I \\ &= \sum_{s=i+1-m-n}^{-1} u(k+s)R_{i,s} + \sum_{s=0}^{i-1} u(k+s/k)R_{i,s} \\ &\qquad + \mu I + N_i + N_i^T - Q_i - \rho_0 I \succcurlyeq 0, \end{aligned} \tag{C9}$$

where the symmetric matrices $R_{i,s}$ are defined as

$$R_{i,s} \triangleq \sum_{j=\max\{0,i-s-n\}}^{\min\{m-1,i-s-1\}} \left[M_{i,j,i-j-s} + M_{i,j,i-j-s}^T\right]. \tag{C10}$$

Because of Eq. 14 we have that $u(k+s) = u(k+s+m)$. Therefore Eq. C9 becomes

$$\left\{ \begin{array}{ll} \displaystyle\sum_{s=0}^{i-1} u(k+s/k)R_{i,s} + \mu I + W_{i,0} \succcurlyeq 0 & \text{if } i \leq m \\ \displaystyle\sum_{s=0}^{m-1} u(k+s/k)R_{i,s} + \sum_{s=0}^{i-m-1} u(k+s/k)R_{i,s} + \mu I + W_{i,0} \succcurlyeq 0 & \text{if } i > m \end{array} \right\}, \tag{C11}$$

where the symmetric matrix $W_{i,0}$ is defined as

$$W_{i,0} \triangleq \sum_{s=i+1-m-n}^{-1} u(k+s)R_{i,s} + N_i + N_i^T - Q_i - \rho_0 I. \tag{C12}$$

Equation C11 can be written as

$$\begin{Bmatrix} \sum_{s=0}^{i-1} u(k+s/k)R_{i,s} + \mu I + W_{i,0} \succcurlyeq 0 & \text{if } i \le m \\ \sum_{s=0}^{i-m-1} u(k+s/k)2R_{i,s} + \sum_{s=i-m}^{m-1} u(k+s/k)R_{i,s} + \mu I + W_{i,0} \succcurlyeq 0 & \text{if } i > m \end{Bmatrix},$$

$$\Leftrightarrow \begin{cases} \sum_{s=0}^{i} x_s W_{i,s} + W_{i,0} & \text{if } i \le m \\ \sum_{s=0}^{i-m-1} x_s V_{i,s} + \sum_{s=i-m}^{m} x_s V_{i,s} + W_{i,0} & \text{if } i > m. \end{cases} \tag{C13}$$

O.E.Δ.

Appendix D: Proof that the Information Matrix M is Singular if $m \le n$

Let $v \in \Re^{n+1}$. Then

$$v^T M v = v^T \left[\sum_{j=0}^{m-1} \lambda^j \phi(k-j+i/k)\phi(k-j+i/k)^T \right] v$$

$$= \sum_{j=0}^{m-1} \lambda^j [\phi(k-j+i/k)^T v]^2 = 0 \tag{D1}$$

$$\Leftrightarrow \phi(k-j+i/k)^T v = 0, \qquad j = 0,1,\ldots,m-1. \tag{D2}$$

If it were $m \le n$, then the number m of homogeneous equations (Eq. D2) would be less than the number $n+1$ of entries of the vector v. Therefore a nonzero vector v would exist that would satisfy Eqs. D2, or, equivalently, Eq. D1. Satisfaction of Eq. D1 for a nonzero v would imply that the matrix M is singular. Consequently, it must be $m > n$, for the matrix M not to be necessarily singular. O.E.Δ.

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